BASICS OF THE DIRAC FIELD

ALLAN LIND JENSEN

Abstract. Founding the physical interpretation of the quantum fields on general relativity is investigated. With the approach of this paper, anti-particle states get positive kinetic energy. Deriving the boundary conditions by variation of the Lagrangian solves the Klein paradox. The possibility of $SU(2,2)$ gauge fields i noted.

1. Notation

The space-time, M, is the 4-dimensional space-time, a Lorentz manifold. The coordinates are denoted by $(x^0, \ldots, x^3) = (t, x, y, z)$, and the coordinate vectors by ∂_t etc. The symbol $\mathbf{d}$ represents the exterior derivative. By $\mathcal{F}(M)$, and TM and ΛM are understood the ring of real scalar fields on M, the vector fields on M, and the covariant vector fields, respectively.

The metric tensor is denoted by $\mathfrak{g} = g_{ij}\ \mathbf{d}x^i \otimes \mathbf{d}x^j$, and the metrically equivalent $(2,0)$ tensor by $\overline{\mathfrak{g}} = g^{ij}\ \partial_i \otimes \partial_j$. Many formulas hold in curved space-time, but formulas for kinetic energy and momentum are restricted to Minkowsky space.

The electro-magnetic field is a 1-form

$$A = -\phi\ \mathbf{d}t + A_x \mathbf{d}x + A_y \mathbf{d}y + A_z \mathbf{d}z \in \Lambda M,$$

where ϕ and $\mathbf{A}$ are the electric and vector potential respectively. The charge-current density is a vector field

$$J = \rho\ \partial_t + J_x\ \partial_x + \ldots \in TM,$$

where ρ is the charge density, and $\mathbf{J}$ the current density. The subscript "f" denotes the metrically equivalent covariant field, e.g. J_f.

Whenever explicit matrices in connexion to the Dirac field are given, the standard representation,

$$\beta = \begin{bmatrix} I & \bigcirc \\ \bigcirc & -I \end{bmatrix}, \text{ and } \gamma^i = \begin{bmatrix} \bigcirc & \sigma_i \\ -\sigma_i & \bigcirc \end{bmatrix},$$

where σ_i are the Pauli spin matrices, is used.

Everything is formulated in classical field theory. This is where the problems and their proposed solutions lie.

2. The relativistic Schrødinger field

Due to its paradigmatic nature, theory of the Klein-Gordon equation is described following [2][1]. The expressions (2) and (5) are the same as in [3], section 5.7, where they are derived from Noether's theorem. In stead of the usual formulas for energy and momentum, gauge invariant expressions (5) and (7) are derived. Using those, the Klein paradox is fully resolved.

[1] See for example [2] exercise 21.2.

The relativistic Schrødinger field is a complex function on M of dimension $[\psi] = \frac{1}{\text{Length}}$ with the Lagrangian

$$\mathcal{L}(\psi) = -\frac{1}{2}\hbar c\ \overline{\mathfrak{g}}\left(\mathbf{d}\bar{\psi} - \frac{iqA}{\hbar}\bar{\psi}, \mathbf{d}\psi - \frac{iqA}{\hbar}\psi\right) - \frac{m^2c^4}{2\hbar c}\ \overline{\psi}\psi$$

(1)

$$= -\frac{1}{2}\hbar c\ \overline{\mathfrak{g}}\left(\mathbf{d}\bar{\psi}, \mathbf{d}\psi\right) + qc\ \overline{\mathfrak{g}}\left(A,\ \mathrm{Im}(\overline{\psi}\mathbf{d}\psi)\right) - \frac{q^2c}{2\hbar}\ \overline{\mathfrak{g}}(A,A)\overline{\psi}\psi - \frac{m^2c^2}{2\hbar}\ \overline{\psi}\psi,$$

where m and q are the mass and charge of the particle associated with the field.

Differentiating the action with respect to the A-field gives the charge-current density[2]

$$J_f = qc\ \mathrm{Im}(\overline{\psi}\ \mathbf{d}\psi) - \frac{q^2cA\overline{\psi}\psi}{\hbar}\ . \tag{2}$$

The stress-energy tensor may be computed by

(3)

$$\mathcal{T}_{ij} = -\left(\tfrac{\partial L}{\partial g^{ij}} + \tfrac{\partial L}{\partial g^{ji}}\right)\ +\ \mathcal{L}\mathfrak{g}_{ij} =$$

$$\hbar c\ \mathrm{Re}\left(\tfrac{\partial\overline{\psi}}{\partial x^i}\tfrac{\partial\psi}{\partial x^j}\right) - qcA_i\ \mathrm{Im}\left(\overline{\psi}\tfrac{\partial\psi}{\partial x^j}\right) - qcA_j\ \mathrm{Im}\left(\overline{\psi}\tfrac{\partial\psi}{\partial x^i}\right) + \tfrac{q^2c}{\hbar}A_iA_j\overline{\psi}\psi\ +\ \mathcal{L}\mathfrak{g}_{ij}.$$

The density of kinetic energy becomes

$$\mathcal{E}_{\mathrm{kin}} = c^2\mathcal{T}^{00} = \frac{\hbar}{c}\ \partial_t\overline{\psi}\partial_t\psi + \frac{2q\phi}{c}\ \ \mathrm{Im}(\overline{\psi}\partial_t\psi) + \frac{q^2\phi^2\overline{\psi}\psi}{\hbar c}\ -\ \mathcal{L}\ . \tag{4}$$

The spacial integral over the Lagrangian vanishes for a solution to the wave equation, so the kinetic energy becomes

$$E_{\mathrm{kin}} = \int \mathcal{E}_{\mathrm{kin}}\ \mathbf{d}V = \int \frac{\hbar}{c^2}\ \partial_t\overline{\psi}\partial_t\psi + \frac{2q\phi}{c}\ \ \mathrm{Im}(\overline{\psi}\partial_t\psi) + \frac{q^2\phi^2}{\hbar c^2}\ \overline{\psi}\psi\ \mathbf{d}V. \tag{5}$$

In the special case of a stationary solution

$$\psi(t,\mathbf{r}) = u(\mathbf{r})e^{-i\omega t},$$

the charge density becomes

$$\rho = \frac{q|u|^2}{\hbar c^2}\ (\hbar\omega\ \ -\ \ q\phi),$$

and the kinetic energy per particle - anti-particles counted negative - reduces to

$$\frac{E_{\mathrm{kin}}q}{\rho} = \hbar\omega - \frac{\int q\phi|u|^2}{\int |u|^2}. \tag{6}$$

Thus the energy per paritcle equals $\hbar\omega$.

The formula for the momentum follows from

$$T_{0x}\ \ =\ \ -\ \frac{\hbar\omega - q\phi}{\hbar}\cdot\Big(\ \ \mathrm{Re}(\overline{u}\cdot(-i\hbar\partial_x u))\ \ -\ \ qA_x|u|^2\Big)\ \ .$$

[2]This is equivalent to applying Noether's theorem with a phase change. Differentiation w.r.t. the gauge field may be a more practical method.

The indices must be raised, because the flow of mass at P crossing a plane perpendicular to λ_P is $\mathcal{T}_P(\lambda_P,\cdot)$, where $\mathcal{T}$ is the (2,0) stress-energy tensor. The momentum per particle becomes

$$\mathbf{p} \;=\; \frac{\int \overline{u}\cdot(-i\hbar\nabla u + q\mathbf{A}u)\ \mathbf{d}V}{\int \overline{u}u\ \mathbf{d}V}\ . \tag{7}$$

2.1. **The Klein Paradox.** In view of [2], section 5.4, it is instructive to review the Klein paradox and study the induced pair production, when a plane wave impinges upon a high potential barrier.

Let the potential barrier be $\phi = V\cdot H(z)$, where H is the Heaviside step function, and let the wave impinging from the negative z-axis be $\psi = Ae^{-i\omega t+ik_l z}$. In the case $qV > mc^2 + \hbar\omega$, a solution

$$\psi(t,x,y,z) \;=\; \begin{cases} Ae^{-i\omega t+ik_l z} \;+\; Be^{-i\omega t-ik_l z} & \text{for} \quad z \leq 0 \\ Ce^{-i\omega t-ik_r z} & \text{for} \quad z \geq 0 \end{cases} \tag{8}$$

exists, where the wave number k_r is the positive solution[3] to $(\hbar\omega - qV)^2 = \hbar^2c^2k_r^2 + m^2c^4$. Continuity of ψ and $\partial_z\psi$ across the boundary gives $B = \frac{k_l+k_r}{k_l-k_r}\,A$ and $C = \frac{2k_l}{k_l-k_r}$. The amplitude of the outgoing wave to the left is greater than the incoming wave. The formulas above should give physically correct values for charge- and current-density, momenta and energy balance.

Only the calculation inside the barrier is interesting. The density is

$$\rho = \frac{q|C|^2}{\hbar c^2}\cdot(\hbar\omega - qV) < 0,$$

and the current density $J_z = -qk_r|C|^2$. This implies a positive velocity

$$v \;=\; -\,\frac{\hbar k_r c^2}{\hbar\omega - qV}. \tag{9}$$

The kinetic energy density by (4) gives

$$\mathcal{E}_{\text{kin}} \;=\; \frac{|C|^2}{\hbar c^2}\,(\hbar\omega - qV)^2\ ,$$

which is positive. The total energy per particle becomes $\hbar\omega$, so that the energy per antiparticle is

$$E \;=\; -\,\hbar\omega.$$

The total energy of a created particle-antiparticle pair vanishes, as it must.

The momentum density becomes

$$\mathcal{P} \;=\; k_r\,\frac{qV - \hbar\omega}{\hbar c^2}|V|^2,$$

which is positive. The momentum per particle is $-\hbar k_r$, so the momentum per antiparticle is $p = \hbar k_r > 0$. Finally, the velocity computed by the formula $v \;=\; \frac{pc^2}{E}$ agrees with (9) above.

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[3]Often, the solution proportional to $e^{-i\omega t+ik_r z}$ is chosen. It represents a situation with two incoming waves, one from each side, with annihilation at $z = 0$ and the amplitudes or potential tuned, so that there is only output to the left.

3. The Dirac Field

The usual starting point is a 4-dimensional representation of the Lorentz group. It turns out that no $\mathbb{C}^4$-representation exists, meaning a homomorphism of the Lorentz group into the group of $\mathbb{C}^4$-isomorphisms, if $\mathbb{C}^4$ endowed with the usual inner product. More precisely, they do exist, but map boosts into the identity.

The well-known representation of the Lorentz group gives transformation matrices, which conserve the scalar product[4]

$$((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = \overline{x}_1 y_1 + \overline{x}_2 y_2 - \overline{x}_3 y_3 - \overline{x}_4 y_4 \ . \tag{10}$$

In this space, the adjoint of a matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* & -C^* \\ -B^* & D^* \end{bmatrix} , \tag{11}$$

where $A, \ldots, D$ are 2 by 2 blocks. The γ-matrices are self-adjoint.

The γ-matrices are introduced as usual by requiring that the vector field γ, with self-adjoint components, solves

$$\gamma \otimes_S \gamma = -\overline{\mathfrak{g}}, \tag{12}$$

where $\otimes_S$ denotes the symmetrized tensor product. The well-known solution in special relativity is

$$\gamma = \frac{1}{c}\beta\ \partial_t + \beta\alpha_x\partial_x + \beta\alpha_y\partial_y + \beta\alpha_z\partial_z =: \sum \sqrt{|g^{ii}|}\gamma^i\partial_i,$$

where β, α are the Dirac matrices.

3.1. Consequences of the form of the Lagrangian.

In this section, a Lagrangian of the form[5]

$$\mathcal{L}(\psi) = -\hbar c\ \mathrm{Im}(\psi, \gamma\psi)_a - mc^2(\psi, \psi)_a + qc(\psi, < \gamma, A > \psi)_a \ , \tag{13}$$

where $(\cdot, \cdot)_a$ is some scalar product, is analyzed.

The wave equation becomes

$$i\hbar c\gamma\psi = mc^2\psi - \frac{i\hbar c}{2}(\mathrm{Div}\ \gamma)\psi + q < A, \gamma > \psi \ . \tag{14}$$

Differentiation with respect to the A-field gives the charge-current density

$$J = qc(\psi, \gamma.\psi)_a \ , \tag{15}$$

where the dot indicates that ψ is a scalar multiplying the vector, γ, no differentiation being involved.

The energy and momentum are still found by (3). The metric tensor occurs implicitly through γ. Therefore, the derivatives of γ with respect to the temporal components of the metric tensor must be found. The present computation stipulates that a diagonal metric tensor leads to

$$\gamma = \sqrt{-\mathfrak{g}_{tt}}\gamma^0\partial_t + \sqrt{\mathfrak{g}_{xx}}\gamma^x\partial_t + \ldots \tag{16}$$

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[4]Physicists have been taking this into account by saying that the invariant bra is $\bar{\psi}\cdot\beta$. But that makes the scalar product representation dependent. Shortly, the γ-matrices will be required to be self-adjoint in the predefined scalar product.

[5] The author only reluctantly proposes symmetrizing the usual expression $(\psi, i\hbar c\gamma\psi)_a + \ldots$ so soon. It forces the Lagrangian density to be real, and results in the correct wave equation (14), which will be essential in section 3.3. It doubles, however, the number of terms when differentiating the action, so the usual form seems quite practical.

It follows immediately that

$$D_{\mathfrak{g}_{00}}\gamma \;=\; -\;\frac{c}{2}\gamma^0\;\partial_t\;, \tag{17}$$

and subsequently the kinetic energy of the wave

$$E_{\text{kin}} = \int (\psi, i\hbar\beta\partial_t\psi - q\phi\gamma^0\psi)_a\;. \tag{18}$$

To find an expression for the momentum, consider the metric tensor

$$\overline{\mathfrak{g}} = -c^{-2}\partial_t\otimes\partial_t + \partial_z\otimes\partial_z + a\,(\partial_t\otimes\partial_z + \partial_z\otimes\partial_t)\;. \tag{19}$$

To use (16) this must be diagonalized; denote the resulting γ-vector by $\overline{\gamma}$. The result is

$$\left.\frac{\partial\overline{\gamma}}{\partial\mathfrak{g}^{tz}} + \frac{\partial\overline{\gamma}}{\partial\mathfrak{g}^{zt}}\right|_{g_{tz}=g_{zt}=0} = \left.\frac{\partial\overline{\gamma}}{\partial a}\right|_{a=0} = \frac{1}{2}\gamma^3\partial_t - \frac{c^2}{2}\gamma^0\partial_z\;. \tag{20}$$

The resulting equation for the momentum is

$$\mathbf{p} = \frac{1}{2}\int (\psi, -i\hbar\beta\nabla\psi - \frac{q\mathbf{A}}{2}\beta\psi)_a + \frac{1}{2c}\int (\psi, \frac{i\hbar}{2c}\overrightarrow{\gamma}.\partial_t\psi - \frac{q\phi}{2c}\overrightarrow{\gamma}.\psi)_a \tag{21}$$

The long honoured expression, $\mathbf{p} = \int (\psi, -i\hbar\beta\nabla\psi)$ originally based on non-relativistic mechanics turns out only to have weight 50%. The physicist, always on the look-out for symmetries, might note that p_z arises from the stress-energy tensor, and because $T^{tz} = T^{zt}$, the expression for $\mathbf{p}$ should exhibit some symmetry in ct and z. The expression (21) is a minimal gauge-invariant expression satisfying this[6].

These results are now applied to stationary flows with the object of finding a consistent theory.

Applying the equations (18) upon a stationary flow, $\psi(t,z) = u(z)e^{-i\omega t}$, leads to $(\psi_1,\phi_2)_a = (\psi_1,\phi_2)$ for particles.

Considering the interaction term for the e^+e^- scattering, it stands to reason that the stationary anti-particle flow must have the form $\psi(t,\mathbf{r}) = u(\mathbf{r})e^{i\omega t}$, where $\hbar\omega = mc^2$. This leads to

$$E = -\hbar\omega(u, c\gamma^0 u)_a,\ \text{(anti-particles)} \tag{22}$$

and

$$\mathbf{p} = \frac{1}{2}(u, -i\hbar\beta\nabla u)_a - \frac{\hbar\omega}{2c}(u, \overrightarrow{\gamma}.u)_a,\ \text{(anti-particles)} \tag{23}$$

First, assume that $(\psi_1,\psi_2)_a = (\psi_1,\psi_2)$. By (22), γ must change sign, when operating on anti-particles. The Dirac equation (14) then implies that an anti-particle at rest has the wrong form $\psi = (0,0,a,b)e^{-i\omega t}$.

Therefore, $(\psi_1,\psi_2)_a = -(\psi_1,\psi_2)$ for anti-particles, and then by the second half of (23) that $(\psi,\gamma.\psi)_a$ is the charge-current density, with negative charge for the anti-particle. This agrees with (15).

[6]Noether's theorem implies that in flat vacuum, $\int(\psi, -i\hbar\beta\nabla\psi)$ is constant. Therefore, $\int(\psi, \frac{i\hbar}{2c}\overrightarrow{\gamma}.\partial_t\psi)$ must be constant as well. This is certainly true for stationary flows.

Table 1. Planar Dirac fields moving to the right

$\begin{pmatrix} \cosh\frac{\zeta}{2} \\ 0 \\ \sinh\frac{\zeta}{2} \\ 0 \end{pmatrix} e^{-i\omega t+ikz}$	$\begin{pmatrix} 0 \\ \cosh\frac{\zeta}{2} \\ 0 \\ -\sinh\frac{\zeta}{2} \end{pmatrix} e^{-i\omega t+ikz}$	$\begin{pmatrix} \sinh\frac{\zeta}{2} \\ 0 \\ \cosh\frac{\zeta}{2} \\ 0 \end{pmatrix} e^{i\omega t-ikz}$	$\begin{pmatrix} 0 \\ -\sinh\frac{\zeta}{2} \\ 0 \\ \cosh\frac{\zeta}{2} \end{pmatrix} e^{i\omega t-ikz}$
$j = \cosh\zeta\partial_t + c\sinh\zeta\partial_z$	$\cosh\zeta\partial_t + c\sinh\zeta\partial_z$	$-\cosh\zeta\partial_t - c\sinh\zeta\partial_z$	$-\cosh\zeta\partial_t - c\sinh\zeta\partial_z$
$S_z = \frac{\cosh\zeta}{2}$	$-\frac{\cosh\zeta}{2}$	$-\frac{\cosh\zeta}{2}$	$\frac{\cosh\zeta}{2}$

3.2. **The resulting theory.** The scalar product[7] of the Dirac field is defined by

$$(\psi_1, \psi_2)_a = \mathrm{sgn}(\psi_1)(\psi_1, \psi_2) \ , \tag{24}$$

where

$$\mathrm{sgn}(\psi) = \mathrm{sign}\ (\psi, \psi), \tag{25}$$

The field γ is defined by (12) with the requirement that the components are self-adjoint under (10).

Symmetry around the z-axis leads to the invariant

$$J_z = \int_V (\psi, i\hbar c x\beta\psi_y)_a - \int_V (\psi, i\hbar c y\beta\psi_z)_a + \int_V (\psi, i\hbar\beta\Omega_{x,y}\psi_y)_a \ ,$$

where

$$\Omega = \frac{1}{4}\gamma_f \wedge \gamma_f$$

is the generator of boosts and rotations. Thus the spin is $S_z = \int_V (\psi, i\hbar\beta\Omega_{x,y}\psi_y)_a$.

Now, it is easy to write solutions to the Dirac equation for particles moving parallel to the z-axis with definite helicity. Se table 1, where $\hbar\omega = mc^2\cosh\zeta$ and $\hbar k = mc\sinh\zeta$; the standard representation being used. The waves move in the same direction as the sign of k. Below each wave: the charge-current density and the spin density.

These four planar waves[8] define the discrete symmetry operators except for a couple of phases each. The first is arbitrary, and the other follows from invariance of the first term of the Lagrangian[9] under the symmetry, which determines the transformation of γ. Parity is $(P\psi)(t, x, y, z) = \beta\psi(t, -x, -y, -z)$, and the time and charge reversal er given by

$$(T\psi)(t, x, y, z) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \overline{\psi(-t, x, y, z)} \tag{26}$$

and

$$(C\psi)(t, x, y, z) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \overline{\psi(t, x, y, z)} \tag{27}$$

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[7]It is not a scalar product in the mathematical sense, because additivity only holds, when the added states as well as their sum have the same sign.

[8]maybe the student should for clarity also write the four with ζ replaced by $-\zeta$ and the same for k.

[9]It can also be determined by rotating the states in table 1 90° around the y-axis, but correct transformations of γ should be checked in any event.

Invariance of the scalar product requires that the matrices associated with P and T are unitary, and that the one associated with C is anti-unitary.

3.3. **Friedmann Geometry.** Let the metric tensor be

$$\mathfrak{g} = -c^2 \mathbf{d}t^2 + R^2(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2) \tag{28}$$

and choose

$$\gamma = \frac{1}{c}\gamma^0 \partial_t + \frac{1}{R(t)}\gamma^1 \partial_x + \dots \tag{29}$$

Its divergence is

$$\text{Div } \gamma = \frac{3R'}{cR}\gamma^0 \ , \tag{30}$$

and the wave equation has a solution

$$\psi(t,x,y,z) = \frac{A}{R^{\frac{3}{2}}(t)} e^{-i\frac{mc^2}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ , \tag{31}$$

which leads to the charge density $\rho = \frac{q|A|^2}{R^3(t)}$.

3.4. **The Klein paradox.** The potential is defined in subsection 2.1. Let the wave impinging from the left be

$$\psi_i(t,z) = A \begin{pmatrix} \cosh(\frac{\zeta}{2}) \\ 0 \\ \sinh(\frac{\zeta}{2}) \\ 0 \end{pmatrix} e^{-i\omega t + ikz} \ , \ z \leq 0 \ , \tag{32}$$

where

$$\hbar\omega = mc^2 \cosh(\zeta)$$

and

$$\hbar k = mc \sinh(\zeta) \ .$$

The only solution with only one wave moving right of the barrier,

$$\psi_t(t,z) = T \begin{pmatrix} \sinh(\frac{\eta}{2}) \\ 0 \\ -\cosh(\frac{\eta}{2}) \\ 0 \end{pmatrix} e^{-i\omega t - i\kappa z} \ , \ z \geq 0 \ , \tag{33}$$

where

$$-\hbar\omega = mc^2 \cosh(\eta) - qV \qquad \hbar\kappa = mc \sinh(\zeta) \ , \tag{34}$$

adds a reflected wave

$$\psi_r(t,z) = R \begin{pmatrix} cosh(\frac{\zeta}{2}) \\ 0 \\ -\sinh(\frac{\zeta}{2}) \\ 0 \end{pmatrix} e^{-i\omega t - ikz} \ , \ z \leq 0 \ . \tag{35}$$

Continuity over the boundary implies

$$R = \frac{A\cosh\left(\frac{\zeta}{2} - \frac{\eta}{2}\right)}{\cosh\left(\frac{\zeta}{2} + \frac{\eta}{2}\right)} \quad \text{and} \quad T = \frac{A\sinh(\zeta)}{\cosh\left(\frac{\zeta}{2} + \frac{\eta}{2}\right)} \ . \tag{36}$$

When $\kappa > 0$, the anti-particle moves to the right, so the incoming wave must induce a pair production. Thus the reflected wave must have greater amplitude than the incoming. But $|R| < |A|$ by (36). This is the Klein paradox, which seems deeply rooted in the kinematics of the Dirac field.

3.5. **Boundary condition when the field changes sign.** It is part of the statement that the theory should be based on the Lagrangian, that the boundary conditions of the fields follow from the condition that the action, $\mathcal{S}$, be stationary. Let $\psi(t,z)$ be a field that is positive for $z < 0$, and negative for $z > 0$. One way to perform the calculation is to compute the directional derivative in the direction to a field $\phi(t,z)$ of the same type.

$$D_\phi \mathcal{S}(\psi) = \lim_{u \to 0} \frac{\mathcal{S}(\psi + \mu\phi) - \mathcal{S}(\psi)}{\mu} =$$

$$-\int (\phi, i\hbar c\gamma\psi)_a - \int \operatorname{Im}(\psi, i\hbar c\gamma\phi)_a - mc^2(\phi,\psi)_a - mc^2(\psi,\phi)_a \ .$$

This must be written as a linear functional of ϕ. The terms involving differentiation with respect to z are

$$i\hbar c\left(\int (\phi, \gamma^z\psi_z)_a + \int (\psi, \gamma^z\phi_z)_a\right) \tag{37}$$

The parenthesis gives

$$\int_{z=-\infty}^{0} (\phi, \gamma^z\psi_z) - \int_{z=0}^{\infty} (\phi, \gamma^z\psi_z) + \int_{z=-\infty}^{0} (\psi, \gamma^z\phi_z) - \int_{z=0}^{\infty} (\psi, \gamma^z\phi_z)$$

Partial integration of the last two terms results in

$$-\int_{z=-\infty}^{0} (\gamma^z\psi_z, \phi) + \int_{z=0}^{\infty} (\gamma^z\psi_z, \phi) + (\psi(0^-), \gamma^z\phi(0^-)) + (\psi(0^+), \gamma^z\phi(0^+))$$

Collecting the terms yields

$$D_\phi\mathcal{S}(\psi) = \int ((i\hbar c\gamma + mc^2)\psi, \phi)_a + \int (\phi, (i\hbar c\gamma + mc^2)\psi)_a +$$
$$(\psi(0^-), \gamma^z\phi(0^-)) + (\psi(0^+), \gamma^z\phi(0^+))$$

The second line[10] is responsible for the boundary condition. Introducing the components of the fields and writing ψ^- for $\psi(0^-)$ and so forth - give

$$\overline{\psi_1^-}\phi_3^- + \overline{\psi_3^-}\phi_1^- + \overline{\psi_1^+}\phi_3^+ + \overline{\psi_3^+}\phi_1^+$$

Assuming continuity across the boundary fails to make this vanish. Assume more generally, that $\psi_1^+ = \alpha\psi_1^-$ and $\psi_3^+ = \beta\psi_3^-$, and analogously for ϕ. The expression reduces to

$$(1 + \overline{\alpha}\beta)(\overline{\phi_1^-}\psi_3^- - \overline{\phi_1^-}\psi_3^-)$$

This vanishes, if $1 + \alpha\overline{\beta} = 0$, a simple choice is $\alpha = 1$ and $\beta = -1$.

[10]It should have been symmetric in ψ and ϕ, and it would have been with the Lagrangian (13).

TABLE 2. Polarized cross sections to lowest order

Process			$\frac{\mathbf{d}\sigma}{\mathbf{d}\Omega}\cdot\frac{4E_{CM}^2}{\alpha^2}$
$\psi(e_R^+)\psi(e_L^-)$	$\to$	$\psi(\mu_R^+)\psi(\mu_L^-)$	$(1+\cos\theta)^2$
		$\psi(\mu_L^-)\psi(\mu_R^+)$	$(1-\cos\theta)^2$
		$\psi(\mu_R^-)\psi(\mu_R^+)$ or $\psi(\mu_L^-)\psi(\mu_L^+)$	$\left(\frac{m_\mu}{E_\mu}\right)^2\sin^2\theta$
$\psi(e_R^+)\psi(e_R^-)$	$\to$	$\psi(\mu_R^-)\psi(\mu_L^+)$ or $\psi(\mu_L^-)\psi(\mu_R^+)$	$\left(\frac{m_e}{E_e}\right)^2\sin^2\theta$
		$\psi(\mu_R^-)\psi(\mu_R^+)$ or $\psi(\mu_L^-)\psi(\mu_L^+)$	$\left(\frac{m_e m_\mu}{E_e E_\mu}\right)^2\cos^2\theta$

The resulting boundary condition

$$\psi_1(0^+)=\psi_1(0^-) \text{ and } \psi_3(0^+)=-\psi_3(0^-) \tag{38}$$

leads to

$$R=\frac{A\cosh\left(\frac{\zeta}{2}+\frac{\eta}{2}\right)}{\cosh\left(\frac{\zeta}{2}-\frac{\eta}{2}\right)} \quad\text{and}\quad T=\frac{A\sinh(\zeta)}{\cosh\left(\frac{\zeta}{2}-\frac{\eta}{2}\right)}\ . \tag{39}$$

This solution has constant current across the boundary, and the energy per anti-particle in the electrostatic field equals $-\hbar\omega$.

4. The polarized cross sections of $e^+e^-\to\mu^+\mu^-$ scattering

The cross sections are computed to the lowest order in [1]. Applying the spinors of table 1, they can readily be computed directly.

Let the rapidity of electron and positron be ζ, the rapidity of the muons be eta, and let the electron move parallel to the z-axis in C.M. system. Then

$$\left(\psi(e_R^+),\gamma.\psi(e_L^-)\right)=\cosh(\zeta)e^{2i\omega t}(\partial_x-i\partial_y) \tag{40}$$

and

$$\left(\psi(e_R^+),\gamma.\psi(e_R^-)\right)=e^{2i\omega t}\partial_z, \tag{41}$$

where the subscripts 'L' and 'R' refer to the helicity state. Rotation the angle θ around the x-axis and replacing ζ with η, gives for the muon pair

$$\left(\psi(\mu_R^+),\gamma.\psi(\mu_L^-)\right)=\cosh(\eta)e^{2i\omega t}(\partial_x-i\cos\theta\partial_y-i\sin\theta\partial_z) \tag{42}$$

The matrix elements follow immediately,

$$\mathcal{M}\left(\psi(e_R^+)\psi(e_L^-)\to\psi(\mu_R^+)\psi(\mu_L^-)\right)=\cosh(\eta)e^{2i\omega t}(1+\cos\theta) \tag{43}$$

etc. From this follows the cross sections of table 2.

5. The Representation gauge fields

The equation (12) has many self-adjoint solutions. A change of representation, $\psi\to G\psi$, where $G(p)=e^{iga(p)A}$ where A is a constant, self-adjoint matrix, g a coupling constant and $a\in\mathcal{F}(M)$ a real function leads to

$$\begin{aligned}\mathrm{Im}(\psi,\gamma\psi) &\mapsto \mathrm{Im}(e^{igaA}\psi,\langle e^{igaA}\gamma e^{-igaA},(e^{-igaA}d(e^{igaA}\psi))\rangle)\\ &= \mathrm{Im}(\psi,\gamma\psi)+\frac{g}{2}(\psi,(\langle\gamma,\mathbf{d}a\ A)\rangle+\langle\mathbf{d}a\ A,\gamma\rangle)\,\psi)\ .\end{aligned}$$

Introduce a dimensionless gauge field

$$D \in \Lambda M \otimes LE =: \Lambda LE \tag{44}$$

which transform by $D \to GDG^* + \mathbf{d}aA$. A more general transformation law is derived in section 6.

The self-interaction term must have the form $\|\mathbf{d}D\|^2$ meaning the double contraction[11] of $\mathbf{d}\overline{D} \otimes \mathbf{d}D$. The natural way to get a scalar from a matrix product is the trace, $(A, B) \to tr(A^*.B)$, and the trace in $\mathbb{C}^4$ with the scalar product (10), follows from contraction of the matrix

$$\mathrm{tr}A = A_{11} + A_{22} - A_{33} - A_{44} \ . \tag{45}$$

Positive semi-definitness fails for this matrix product. It is reasonable to believe that the normal adjoint and trace of a matrix must be used. This results in a Lagrangian

$$\mathcal{L} = -\frac{\hbar c}{2}\|\mathbf{d}D\|^2 + \frac{\hbar c g}{2}(\psi, (\langle\gamma, D\rangle + \langle D, \gamma\rangle)\,\psi)_a \ . \tag{46}$$

Let a Lorentz transformation transform a Dirac field $\psi \mapsto U\psi$ and a 1-form by $\lambda \mapsto u\lambda$. Then the transformed M-field is given by

$$M' = u(UM.U^*) \ . \tag{47}$$

Assume that the value of a field, D, at one point x is $D_x = M_{t,x}\mathbf{d}t$. When observed from a rotated observer it has the form $D^R_p = e^{iR\theta}M_{t,x}e^{-iR\theta}\mathbf{d}t$, and observed from an oberserver moving in the z-direction with velocity $v = -c\tanh(\zeta)$, the form $D^B_p = \cosh(\zeta)e^{iB_z\zeta}M_{t,x}e^{-iB_z\zeta}\mathbf{d}t + \sinh(\zeta)e^{iB_z\zeta}M_{t,x}e^{-iB_z\zeta}\mathbf{d}z$. Expanding the exponentials results in

$$M^R = \left(\cos^2\left(\frac{\theta}{2}\right)M + i\sin(\theta)[R, M] + \sin^2\left(\frac{\theta}{2}\right)RMR\right)\mathbf{d}t$$

and

$$\begin{aligned} M^B &= \left(\cosh^2\left(\frac{\zeta}{2}\right)M + i\sinh(\zeta)[B_z, M] + \sinh^2\left(\frac{\zeta}{2}\right)BMB\right)\cosh(\zeta)\mathbf{d}t \\ &+ \left(\cosh^2\left(\frac{\zeta}{2}\right)M + i\sinh(\zeta)[B_z, M] + \sinh^2\left(\frac{\zeta}{2}\right)BMB\right)\sinh(\zeta)\mathbf{d}z \ . \end{aligned}$$

This indicates that the fields $i[R, M]\mathbf{d}t$ as well as $i[B, M]\mathbf{d}t$ and $M\mathbf{d}z$ are of the same physical nature as $M\mathbf{d}t$. Analogously for rotations. This indicates that the fields are divided into a few families with the same physics.

The transformation proporties of the 16 self-adjoint matrices generated from γ^i are well-known. The identity matrix is invariant under Lorentz transformations, but also

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \ . \tag{48}$$

It turns out that the Lagrangian already is invariant with respect to change of representation generated by γ^5, because the interaction term vanishes,

$$\left\langle\gamma, \gamma^5\right\rangle + \left\langle\gamma^5, \gamma\right\rangle = 0 \ . \tag{49}$$

[11] In coordinates written $\overline{\mathbf{d}D}^{u,v}\mathbf{d}D_{u,v}$.

TABLE 3. Families of representation fields

Familiy	Symbol	Generators
A field	A	$\frac{1}{2}I$
γ^5 field	-	$\frac{1}{2}\gamma^5$
γ fields	Γ	$\frac{1}{2}\gamma^i$
ξ fields	Ξ	$\frac{1}{2}\gamma^5\gamma^i$
Generator fields	G	S_x, S_y, S_z, B_x, B_y, B_z

TABLE 4. The Three distinguished fields

Form	with	dimension	interaction	P	C	T
$\frac{1}{2}AI$	$A \in \Lambda M$	$[A] = 1$	$gc(\psi, \langle A, \gamma\rangle\, \psi)_a$	+	-	-
$\frac{1}{2}\lambda\gamma^5$	$\lambda \in \Lambda M$	$[\lambda] = 1$	0	-	+	+
$\frac{1}{2}f\gamma_f$	$f \in \mathcal{F}M$	$[f] = \text{Length}^{-1}$	$fg_\gamma c(\psi, \psi)_a$	+	-	-
$\frac{1}{2}f\xi_f$	$f \in \mathcal{F}M$	$[f] = \text{Length}^{-1}$	0	-	+	+

The fields generated by γ^i form one family; another is generated by the generators of rotations and boosts. The four self-adjoint matrices, $\xi^i := i\gamma^5\gamma^i$, form the last family, see table 3.

The space of gauge fields can be split up into two commuting spaces, span ($\frac{1}{2}I$ and $SU^*(2,2)$ that is generated by the forteen other generators of table 3. Their coupling constants may be different.

Among the possible fields, three have been found, which are invariant under Lorentz transformations. They are listed in the table 4. The functions and covariant vector fields must be real, so that the matrices remain self-adjoint. The signs under the two discrete Lorentz isometries, parity and time inversion, P and T, and charge inversion C, make the interaction term invariant. (Gælder det også Ξ feltet?)

A note on Noether's theorem may be in place. The transformations generated by I and γ^5 are the only ones leaving γ - and therefore the form of the Lagrangian - unchanged. Only for these are Noether's theorem applicable implying conservation of electric charge and nought.

6. Transformation of Gauge fields

This work builds on section 12.1 of [3]. A gauge transformation $G = e^{igM}$, where M is a self-adjoint matrix field and g the coupling constant, and a minimal interaction introduced by the covariant derivative $D\psi = \mathbf{d}\psi + igB$ leads to the transformation of the vector field B given by the first part of (12.5) of [3],

$$B \mapsto \widehat{G}(B) = GBG^* + \frac{i}{g}\,(\mathbf{d}G)G^* \;. \tag{50}$$

The transformed field is easily seen to be self-adjoint. It must also be shown that

$$\widehat{G_1 G_2} = \widehat{G_1}\widehat{G_2} \;, \tag{51}$$

which follows from the product rule.

The second line of (12.5) - although widely accepted, and being true in 1D - seems to fail in general. The transformation law derived in the next section holds for $SU(2)$ and $SU(2,2)$.

6.1. **Explicit formula for pairwise anti-symmetric generators.** Assume that the gauge group have self-adjoint generators T_i, with

$$\{T_i, T_j\} = 2\epsilon_i^2 \delta_{ij} I \ . \tag{52}$$

In $SU(2)$ with generators $\frac{1}{2}\sigma$, the values are $\epsilon_i = \frac{1}{2}$; in $SU^*(2,2)$ the values are $\epsilon_i = \frac{1}{2}$ or $\epsilon_i = \frac{i}{2}$, seven of each. Write

$$A = \sum a_i T_i, \tag{53}$$

where $a_i \in \mathcal{F}(M)$ are real functions, and introduce

$$r := \sqrt{\sum \epsilon_i^2 a_i^2} \quad \text{and} \quad \lambda = \sum \epsilon_i^2 a_i \mathbf{d} a_i \ , \tag{54}$$

so that

$$\mathbf{d}r = \frac{1}{r}\ \lambda \ . \tag{55}$$

First, an closed expression for G is found. Applying

$$A^2 = \sum a_i a_j T_i T_j = r^2 \tag{56}$$

leads to

$$\begin{aligned} G &= \sum \frac{(igA)^n}{n!} = \sum \frac{(-)^n (gr)^{2n}}{(2n)!} I + igA \sum \frac{(-)^n (gr)^{2n}}{(2n+1)!} \\ &= \cos(gr) I + i\ \frac{\sin(gr)}{r} A \ . \end{aligned} \tag{57}$$

The external derivative of G^* becomes

$$\mathbf{d}G^* = -\frac{g\sin(gr)}{r}\lambda + i\frac{\sin(gr) - gr\cos(gr)}{r^3} A\lambda - i\frac{\sin(gr)}{r}\mathbf{d}A \ .$$

In the expression for $(\mathbf{d}G^*)G$, the term $(\mathbf{d}A)A$ may be rewritten

$$(\mathbf{d}A)A = \lambda I + \frac{1}{2}\sum_{i<j} (a_j \mathbf{d}a_i - a_i \mathbf{d}a_j)[T_i, T_j] \ . \tag{58}$$

Now, $(\mathbf{d}G^*)G$ can be reduced leading to

$$\begin{aligned} (\mathbf{d}G^*)G &= i\ \frac{\cos(gr)\sin(gr) - gr}{r^3} A\lambda - i\ \frac{\cos(gr)\sin(gr)}{r}\mathbf{d}A \\ &\quad + \frac{1}{2}\sum_{i<j}(a_j \mathbf{d}a_i - a_i \mathbf{d}a_j)[T_i, T_j] \ . \end{aligned} \tag{59}$$

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[3] Alexia L. Rebenko. *Theory of Interacting Quantum Fields*. De Gruyter, 2012.